A General Approach to the Study of Chebyshev Subspaces in L_1-Approximation of Continuous Functions

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INTRODUCTION

In the last ten years, the question of uniqueness of best L_1-approximation of continuous functions has been widely investigated. The increasing research activity in this area was inspired by the fact that uniqueness of best L_1-approximation of continuous functions imposes less restrictions on the approximating family than in the case of Chebyshev approximation. In the present paper we shall consider the problem of characterizing those subspaces of continuous functions which guarantee unicity of best L_1-approximation with respect to all positive weights. This problem will be studied in the general context of Banach space valued functions. Some applications of the main results will also be discused.

Notation. Let K be a compact subset of $\mathbb{R}^n$ ($n \geq 1$) such that $K = \overline{\text{Int } K}$ and $\mu(\text{int } K) > 0$, where $\mu(...)$ denotes the Lebesgue measure in $\mathbb{R}^n$. Furthermore, let X be a real Banach space with norm $\|\cdot\|_X$. W denotes the set of all measurable real functions ω on K such that $0 < \inf\{\omega(x): x \in K\} \leq \sup\{\omega(x): x \in K\} < \infty$. Consider the space $C(K, X)$ of continuous functions $f: K \to X$. Given a weight $\omega \in W$ we introduce the norm

$$\|f\|_\omega = \int_K \omega(x) \|f(x)\|_X \, d\mu \qquad (f \in C(K, X))$$

and denote by $C_\omega(K, X)$ the space $C(K, X)$ endowed with the above norm.

Let now M be a finite-dimensional subspace of $C_\omega(K, X)$. As usual we say that $p \in M$ is a best approximant of $f \in C_\omega(K, X)$ if and only if $\|f - p\|_\omega = \inf\{\|f - q\|_\omega : q \in M\}$. In the present paper we shall study the

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unicity of best approximation in $C_\omega(K, X)$. The existence of best approximants follows immediately since M is finite-dimensional. The subspace M is called a Chebyshev subspace of $C_\omega(K, X)$ if each $f \in C_\omega(K, X)$ has a unique best approximant in M. Let us denote by

$$\tau_X(u, v) = \lim_{t \to 0} \frac{\|u + tv\|_X - \|u\|_X}{t}$$

the left derivative of norm $\|\cdot\|_X (u, v \in X, u \neq 0)$. It is well-known that this limit always exists, the functional $\tau_X(u, \cdot)$ is frequently used in approximation theory.

We shall need the following characterization of best approximation (see [15]). The element $p \in M$ is a best approximant of $f \in C_\omega(K, X)$ if and only if for any $q \in M$

$$\int_{K \setminus Z(f-p)} \omega(x)\, \tau_X(f-p, q)(x)\, d\mu \leqslant \int_{Z(f-p)} \omega(x)\|q(x)\|_X\, d\mu. \tag{1}$$

Here and throughout the paper $Z(g) = \{x \in K : g(x) = 0\}$.

GENERAL THEORY

In this section we shall give some general theorems on unicity of best approximation in $C_\omega(K, X)$.

Given the linear subspace $M \subset C(K, X)$ we set $M^* = \{q^* \in C(K, X) :$ there exists $\tilde{q} \in M$ such that for each $x \in K$ either $q^*(x) = \tilde{q}(x)$ or $q^*(x) = -\tilde{q}(x)\}$. This notation originates from [11]. The next theorem gives a useful criteria for Chebyshev subspaces in $C_\omega(K, X)$. For the case when $X = \mathbb{R}$ it was proved by Strauss [12]. In [4] we verified it for $X = \mathbb{R}^k$ endowed with the Euclidean norm.

THEOREM 1. *Let M be a finite dimensional subspace of $C_\omega(K, X)$, $\omega \in W$. Then in order that M be a Chebyshev subspace of $C_\omega(K, X)$ it is necessary that no $q^* \in M^* \setminus \{0\}$ has 0 as a best approximant in M. Moreover, if the Banach space X is strictly convex then this condition is also sufficient.*

Proof. Let us verify the necessity. If 0 is a best approximant of some $q^* \in M^* \setminus \{0\}$ then (1) holds with $f = q^*$ and $p = 0$. Furthermore there exists a $\tilde{q} \in M \setminus \{0\}$ such that $q^*(x) = \gamma(x)\, \tilde{q}(x)$ where $\gamma(x)$ is either 1 or -1 ($x \in K$). Evidently, we have $Z(q^* + \delta\tilde{q}) = Z(q^*)$ if $-1 < \delta < 1$. Moreover, since $\tau_X(\alpha u, v) = \tau_X(u, v)$ for any $u, v \in X$, $u \neq 0$ and $\alpha > 0$ it follows that for any $x \in K \setminus Z(q^*)$, and $q \in M$ $\tau_X(q^* + \delta\tilde{q}, q)(x) = \tau_X((\gamma + \delta)\, \tilde{q}, q)(x) = \tau_X(q^*, q)(x)$. Thus by (1) $-\delta\tilde{q} \in M \setminus \{0\}$ is a best approximant of q^*, as well.

Let us prove now that if X is strictly convex then the condition of theorem is sufficient. Assume that $f \in C_\omega(K, X)$ has two different best approximants $p_1, p_2 \in M$. Then $(p_1 + p_2)/2$ is also a best approximant, hence almost everywhere at K

$$\|(f - p_1) + (f - p_2)\|_X = \|f - p_1\|_X + \|f - p_2\|_X \tag{2}$$

holds. By continuity of the functions involved and the relation $K = \overline{\operatorname{Int} K}$ we obtain that (2) holds for each $x \in K$. Now using the strict convexity of X we can conclude that for every $x \in K$ either one of the quantities $(f - p_1)(x)$ and $(f - p_2)(x)$ is zero or $(f - p_1)(x) = c(f - p_2)(x)$, where $c = c(x) \in \mathbb{R} \setminus \{0\}$. Then setting $f^*(x) = f(x) - (p_1(x) + p_2(x))/2$ we obtain that for any $x \in K \setminus Z(p_1 - p_2)$

$$f^*(x) = \gamma(x)(p_1 - p_2)(x), \tag{3}$$

where $\gamma(x)$ is a real constant. Moreover (2) implies that $Z(f^*) \subset Z(p_1 - p_2)$, hence $\gamma(x) \neq 0$ if $x \in K \setminus Z(p_1 - p_2)$. Let us consider p^* given by

$$p^*(x) = \frac{\|(p_1 - p_2)(x)\|_X}{\|f^*(x)\|_X} f^*(x)$$

if $x \in K \setminus Z(f^*)$ and $p^*(x) = 0$ for $x \in Z(f^*)$. Since p^* is continuous at $K \setminus Z(f^*)$ and $Z(f^*) \subset Z(p_1 - p_2)$ it follows that $p^* \in C(K, X)$. Moreover by (3) for $x \in K \setminus Z(p_1 - p_2)$ we have $p^*(x) = (p_1 - p_2)(x) \operatorname{sign} \gamma(x)$. Thus $p^* \in M^* \setminus \{0\}$. Furthermore, using again (3) we have for any $q \in M$ and $x \in K \setminus Z(p_1 - p_2)$

$$\tau_X(f^*, q)(x) = \tau_X(\gamma(p_1 - p_2), q)(x) = \tau_X(p^*, q)(x). \tag{4}$$

Finally, taking into account that 0 is a best approximant of f^* in M we derive by (1) and (4) that for each $q \in M$

$$\int_{K \setminus Z(p^*)} \omega(x)\, \tau_X(p^*, q)(x)\, d\mu$$

$$= \int_{K \setminus Z(p^*)} \omega(x)\, \tau_X(f^*, q)(x)\, d\mu$$

$$\leqslant \int_{K \setminus Z(f^*)} \omega(x)\, \tau_X(f^*, q)(x)\, d\mu + \int_{Z(p^*) \setminus Z(f^*)} \omega(x)|\tau_X(f^*, q)(x)|\, d\mu$$

$$\leqslant \int_{Z(f^*)} \omega(x)\|q(x)\|_X\, d\mu + \int_{Z(p^*) \setminus Z(f^*)} \omega(x)\|q(x)\|_X\, d\mu$$

$$= \int_{Z(p^*)} \omega(x)\|q(x)\|_X\, d\mu.$$

(In the last inequality we have used the obvious relation $|\tau_X(u, v)| \leqslant \|v\|_X$, $u, v \in X, u \neq 0$.) Thus 0 is a best approximant of $p^* \in M^* \backslash \{0\}$. The proof of the theorem is completed.

Theorem 1 reduces the study of L_1-approximation of functions in $C_\omega(K, X)$ to M^* but it is not very convenient for concrete applications. We shall now introduce an L_1-norm independent property of M which turns out to be very useful in the study of the uniqueness of L_1-approximation.

DEFINITION. The finite dimensional subspace $M \subset C(K, X)$ is called an A-space (or is said to satisfy the A-property) if for any $p^* \in M^* \backslash \{0\}$ there exists a $p \in M$ such that

   (i) $p = 0$ a.e. on $Z(p^*)$

   (ii) $\tau_X(p^*, p)(x) \geqslant 0$ a.e. at $K \backslash Z(p^*)$ and

this inequality is strict on a subset of $K \backslash Z(p^*)$ of positive measure.

The notation of A-spaces in the case when $X = \mathbb{R}$, $K = [a, b]$ first appeared in a paper by Strauss [11], who attributes it to an oral communication of DeVore.

Strauss [11], in the above case, also proved this next result which is an easy consequence of Theorem 1 and the above definition.

THEOREM 2. *Let X be a strictly convex Banach space and assume that M is an A-subspace of $C(K, X)$. Then M is a Chebyshev subspace of $C_\omega(K, X)$ for every $\omega \in W$.*

Proof. If our claim fails to hold for some $\tilde{\omega} \in W$ then by Theorem 1 there exists a $p^* \in M^* \backslash \{0\}$ such that

$$\int_{K \backslash Z(p^*)} \tilde{\omega}(x) \tau_X(p^*, q)(x) \, d\mu \leqslant \int_{Z(p^*)} \tilde{\omega}(x) \|q(x)\|_X \, d\mu$$

for any $q \in M$. On the other hand the A-property of M ensures the existence of a $p \in M$ for which the left side of the above inequality is strictly positive while the right side is 0.

Thus by Theorem 2 if X is strictly convex then the A-property implies that uniqueness holds with respect to each weight $\omega \in W$. It turns out that assuming that X is smooth this statement can be reversed. Recall that X is smooth if at every point of its unit sphere there exists a unique tangent functional. It is known that in this case $\tau_X(u, \cdot)$ is a linear functional.

THEOREM 3. *Let X be a smooth Banach space and assume that M is a Chebyshev subspace of $C_\omega(K, X)$ for each $\omega \in W$. Then M satisfies the A-property.*

Proof. It follows by Theorem 1 and (1) that for any $q^* \in M^* \setminus \{0\}$ and $\omega \in W$ there exists $q \in M$ for which

$$\int_{K \setminus Z(q^*)} \omega(x)\, \tau_X(q^*, q)(x)\, d\mu > \int_{Z(q^*)} \omega(x) \|q(x)\|_X\, d\mu. \tag{5}$$

Let $q^* \in M^* \setminus \{0\}$ be given and set $\tilde{M} = \{q \in M \colon q = 0 \text{ a.e. on } Z(q^*)\}$. Evidently, $\tilde{M}$ is a nonempty linear subspace of M.

Our main goal is to prove the following:

Claim. There exists a $q_0 \in \tilde{M}$ such that

$$\int_{K \setminus Z(q^*)} \omega(x)\, \tau_X(q^*, q_0)(x)\, d\mu \neq 0 \tag{6}$$

for any $\omega \in W$.

Assume that our claim is false, i.e., for any $q \in \tilde{M}$ we can find an $\omega \in W$ satisfying

$$\int_{K \setminus Z(q^*)} \omega(x)\, \tau_X(q^*, q)(x)\, d\mu = 0. \tag{7}$$

Let dim $\tilde{M} = k$, $k \geqslant 1$, and let $q_1, \ldots, q_k$ be a basis in $\tilde{M}$. It is well known that in a smooth Banach space X the functional $\varphi(v) = \tau_X(u, v)$ is linear for any fixed $u \in X$, $u \neq 0$. Therefore it follows by (7) that for any $\bar{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k$ there exists an $\omega \in W$ such that

$$0 = \int_{K \setminus Z(q^*)} \omega(x)\, \tau_X\left(q^*, \sum_{i=1}^{k} b_i q_i\right)(x)\, d\mu$$

$$= \sum_{i=1}^{k} b_i \int_{K \setminus Z(q^*)} \omega(x)\, \tau_X(q^*, q_i)(x)\, d\mu. \tag{8}$$

Consider the set

$$A_0 = \left\{ \left(\int_{K \setminus Z(q^*)} \omega(x)\, \tau_X(q^*, q_i)(x)\, d\mu \right)_{i=1}^{k} \colon \omega \in W \right\}.$$

Obviously, A_0 is a convex subset of $\mathbb{R}^k$. Moreover by (8), A_0 has nonempty intersection with any hyperplane $H(\bar{b}) = \{\bar{a} \in \mathbb{R}^k \colon \langle \bar{a}, \bar{b} \rangle = 0\}$, $\bar{b} \in \mathbb{R}^k (\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^k$). Assume that A_0 is an r-dimensional convex subset of $\mathbb{R}^k$. Let us prove that $\bar{0} \in A_0$. If $r = 0$ this holds trivially hence we may assume that $1 \leqslant r \leqslant k$. Since $\bar{0}$ is a cluster point of A_0 we can conclude that A_0 contains r linearly independent vectors, i.e., for some $\omega_1, \ldots, \omega_r \in W$

$$\bar{1}_j = \left(\int_{K \setminus Z(q^*)} \omega_j(x)\, \tau_x(q^*, q_i)(x)\, d\mu \right)_{i=1}^{k}, \qquad 1 \leqslant j \leqslant r,$$

are linearly independent. We state that A_0 is an open subset of the flat $F_r = \mathrm{span}\{\bar{1}_1, ..., \bar{1}_r\}$. Consider an arbitrary $\bar{c} \in A_0$, i.e., for some $\omega_{\bar{c}} \in W$

$$\bar{c} = (c_i)_{i=1}^{k} = \left(\int_{K \setminus Z(q^*)} \omega_{\bar{c}}(x)\, \tau_x(q^*, q_i)(x)\, d\mu \right)_{i=1}^{k}. \tag{9}$$

Evidently, if we choose $h > 0$ to be small enough then $\omega_{\bar{c}} \pm h\omega_j \in W$ ($j = 1, ..., r$). Therefore

$$\bar{\alpha}_{j,\gamma} = \left(\int_{K \setminus Z(q^*)} (\omega_{\bar{c}} + \gamma h\omega_j)(x)\, \tau_x(q^*, q_i)(x)\, d\mu \right)_{i=1}^{k} \in A_0 \tag{10}$$

for any $1 \leqslant j \leqslant r$ and $\gamma = \pm 1$. Furthermore, by (9) and (10)

$$\bar{\alpha}_{j,\gamma} = \bar{c} + \gamma h\bar{1}_j \ (1 \leqslant j \leqslant r, \gamma = \pm 1). \tag{11}$$

Moreover convexity of A_0 implies that for any $\tau_{j,\gamma} \geqslant 0$ $(1 \leqslant j \leqslant r, \gamma = \pm 1)$ such that $\sum_{j=1}^{r} \tau_{j,1} + \sum_{j=1}^{r} \tau_{j,-1} = 1$ we have by (11)

$$\sum_{j=1}^{r} \tau_{j,1}\bar{\alpha}_{j,1} + \sum_{j=1}^{r} \tau_{j,-1}\bar{\alpha}_{j,-1} = \bar{c} + h \sum_{j=1}^{r} (\tau_{j,1} - \tau_{j,-1})\bar{1}_j \in A_0.$$

This and linear independence of $\bar{1}_j$, $1 \leqslant j \leqslant r$, yield that A_0 contains an r-dimensional ball with center at $\bar{c}$. Thus A_0 is an open convex subset of F_r. If $\bar{0} \notin A_0$ then $\bar{0} \in BdA_0$, and there exists a hyperplane $\tilde{F}$ in F_r supporting A_0 at $\bar{0}$. Since A_0 intersects any hyperplane $H(\bar{b}) = \{\bar{a} \in \mathbb{R}^k : \langle \bar{a}, \bar{b} \rangle = 0\}$ it should intersect $\tilde{F}$, as well. But this contradicts the fact that A_0 is open in F_r. Therefore $\bar{0}$ should necessarily belong to A_0, i.e., for some $\bar{\omega} \in W$

$$\int_{K \setminus Z(q^*)} \bar{\omega}(x)\, \tau_x(q^*, q_i)(x)\, d\mu = 0 \ (1 \leqslant i \leqslant k).$$

This and linearity of $\tau_x(u, \cdot)$ $(u \neq 0)$ imply that

$$\int_{K \setminus Z(q^*)} \bar{\omega}(x)\, \tau_x(q^*, q)(x)\, d\mu = 0, \qquad q \in \tilde{M}. \tag{12}$$

Let now $q_1, ..., q_n$ be a basis in M, where as above $q_1, ..., q_k$ $(k \leqslant n)$ is a basis in $\tilde{M}$ and set $M' = \mathrm{span}\{q_{k+1}, ..., q_n\}$. Consider the functionals

$$\eta_1(q) = \int_{Z(q^*)} \|q(x)\|_X\, d\mu, \qquad \eta_2(q) = \sup_{x \in K} \|q(x)\|_X, \qquad q \in M'.$$

Obviously, $\eta_2(q)$ is a norm on M'. Moreover, since $\eta_1(q) > 0$ for $q \in M' \setminus \{0\}$, $\eta_1(q)$ is a norm on M', too. By the equivalence of norms in finite dimensional spaces we obtain that there exists a positive constant ξ independent of $q \in M'$ such that $\eta_2(q) \leqslant \eta_1(q)\, \xi$ for every $q \in M'$. Consider now the weight $\omega^* \in W$ given by

$$\omega^*(x) = \begin{cases} \bar{\omega}(x), & x \in K \setminus Z(q^*) \\ \xi \sup_{x \in K} \bar{\omega}(x)\, \mu(K), & x \in Z(q^*). \end{cases}$$

Then by (12) and linearity of τ_x-functional for any $q = \bar{q}_1 + \bar{q}_2 \in M$, where $\bar{q}_1 \in \tilde{M}$, $\bar{q}_2 \in M'$, we have

$$\int_{K \setminus Z(q^*)} \omega^*(x)\, \tau_x(q^*, q)(x)\, d\mu$$

$$= \int_{K \setminus Z(q^*)} \bar{\omega}(x)\, \tau_x(q^*, \bar{q}_2)(x)\, d\mu$$

$$\leqslant \sup_{x \in K} \bar{\omega}(x)\, \mu(K)\, \eta_2(\bar{q}_2) \leqslant \xi \sup_{x \in K} \bar{\omega}(x)\, \mu(K)\, \eta_1(\bar{q}_2)$$

$$= \int_{Z(q^*)} \omega^*(x) \| \bar{q}_2(x) \|_x\, d\mu = \int_{Z(q^*)} \omega^*(x) \| q(x) \|_x\, d\mu.$$

But this contradicts (5).

By this contradiction we obtain that our claim is true, i.e., there exists a $q_0 \in \tilde{M}$ satisfying (6) for all $\omega \in W$. This implies that either $\tau_x(q^*, q_0) \geqslant 0$ a.e. on $K \setminus Z(q^*)$ or $\tau_x(q^*, q_0) \leqslant 0$ a.e. on $K \setminus Z(q^*)$. Indeed, if we assume that the sets

$$S_i = \{ x \in K \setminus Z(q^*) : (-1)^i\, \tau_x(q^*, q_0)(x) > 0 \}, \qquad i = 1, 2,$$

have both positive measures then

$$(-1)^i \int_{S_i} \tau_x(q^*, q_0)(x)\, d\mu > 0 \qquad (i = 1, 2).$$

Choosing $\varepsilon > 0$ to be sufficiently small and setting for $i = 1, 2$

$$\omega_i(x) = \begin{cases} 1, & x \in S_i \\ \varepsilon, & x \in K \setminus S_i \end{cases}$$

we obtain

$$(-1)^i \int_{K \setminus Z(q^*)} \omega_i(x)\, \tau_x(q^*, q_0)(x)\, d\mu > 0 \qquad (i = 1, 2). \tag{13}$$

Furthermore $\beta\omega_1 + (1 - \beta)\,\omega_2 \in W$ for any $0 \leqslant \beta \leqslant 1$ hence by (13) for some $0 < \beta^* < 1$

$$\int_{K \setminus Z(q^*)} (\beta^*\omega_1 + (1 - \beta^*)\,\omega_2)(x)\,\tau_X(q^*, q_0)(x)\,d\mu = 0.$$

But this contradicts (6). Thus we may assume without loss of generality by linearity of τ_X-functional that $\tau_X(q^*, q_0)(x) \geqslant 0$ a.e. at $K \setminus Z(q^*)$ and by (6) this inequality should be strict on a subset of $K \setminus Z(q^*)$ of positive measure. In addition, $q_0 \in \tilde{M}$, i.e., $q_0 = 0$ a.e. $Z(q^*)$. Thus we have found an element in M required by the A-property. The theorem is proved.

The next statement is an immediate consequence of Theorems 2 and 3.

COROLLARY 1. *Let X be a strictly convex smooth Banach space and let M be a finite dimensional subspace of $C(K, X)$. Then in order that M be a Chebyshev subspace of $C_\omega(K, X)$ for all weights $\omega \in W$ it is necessary and sufficient that M satisfies the A-property.*

In case when $X = \mathbb{R}$, $K = [a, b]$ Theorem 3 was verified by the author [5]. In an independent work Pinkus [10] gave another version of this result for $X = \mathbb{R}$, $K = [a, b]$. Imposing a slight restriction on M, i.e. $\mu(Z(q)) = \mu(\text{Int } Z(q))$, $\forall q \in M$ he showed that the result remains true even if only continuous weights are considered. It can be shown that with the same restriction on M Theorem 3 also holds for any smooth Banach space X, if we replace the set of measurable bounded weights by continuous weights. Moreover, in case when $X = \mathbb{R}$, $K = [a, b]$ we can improve the theorem further considering only the set W^∞ of positive infinitely differentiable weights at $[a, b]$. Let us outline the proof.

THEOREM 4. *Let M be a finite dimensional subspace of $C([a, b], \mathbb{R})$ with the property that $\mu(Z(q)) = \mu(\text{Int } Z(q))$ for any $q \in M$. If M is a Chebyshev subspace of $C_\omega([a, b], \mathbb{R})$ for any $\omega \in W^\infty$ then M is an A-space.*

Proof. First of all let us note that $\tau_\mathbb{R}(u, v) = v \operatorname{sign} u$ $(u, v \in \mathbb{R}, u \neq 0)$. The proof is identical to the proof of Theorem 3 until we get a weight $\bar\omega \in W^\infty$ satisfying (12). The only properties of the set of weights W^∞ needed in this part of the proof are the following: (i) W^∞ is a convex cone, i.e., $\alpha\omega_1 + \beta\omega_2 \in W^\infty$ for any $\omega_1, \omega_2 \in W^\infty$ and $\alpha, \beta > 0$; (ii) for every $\omega_1, \omega_2 \in W^\infty$ we can choose $\alpha > 0$, to be small enough so that $\omega_1 - \alpha\omega_2 \in W^\infty$. Furthermore we again let $q_1, ..., q_n$ be a basis in M such that $q_1, ..., q_k$ $(k \leqslant n)$ is a basis of $\tilde{M}$ and we set $M' = \operatorname{span}\{q_{k+1}, ..., q_n\}$. Evidently, no $q \in M' \setminus \{0\}$ can vanish at Int $Z(q^*)$ since otherwise the relation $\mu(Z(q^*)) = \mu(\text{Int } Z(q^*))$ would imply that q vanishes a.e. on $Z(q^*)$. By a simple compactness argument we can derive the existence of a

finite number of closed intervals $[\alpha_j, \beta_j] \subset \text{Int } Z(q^*)$ $(1 \leqslant j \leqslant s)$ such that no $q \in M' \setminus \{0\}$ vanishes on all of them. Furthermore for any constant $R > 0$ we can construct $\omega^* \in W^\alpha$ such that $\omega^* = \bar{\omega}$ at $[a, b] \setminus Z(q^*)$ and $\omega^* = R$ at $\bigcup_{j=1}^s [\alpha_j, \beta_j]$. This can be easily done.

The rest of the proof can now be completed following the proof of Theorem 3.

EXAMPLE 1.   Set $X = \mathbb{R}_p^k$, where $\mathbb{R}_p^k$ denotes the space $\mathbb{R}^k$ endowed with the l_p-norm $\|a\|_p = (\sum_{i=1}^k |a_i|^p)^{1/p}$ $(a = (a_1, ..., a_k) \in \mathbb{R}^k)$ and $1 < p < \infty$. (Note that R_2^2 is equivalent to $\mathbb{C}$). Then we have for $u = \{u_i\}_{i=1}^k$, $v = \{v_i\}_{i=1}^k \in \mathbb{R}_p^k$, $u \neq 0$

$$\tau_{\mathbb{R}_p^k}(u, v) = \|u\|_p^{1-p} \sum_{i=1}^k |u_i|^{p-1} v_i \, \text{sign } u_i$$

$$= \|u\|_p^{1-p} \sum_{u_i \neq 0} |u_i|^{p-1} \tau_{\mathbb{R}}(u_i, v_i).$$

This relation immediately implies that if $M_1, ..., M_k \subset C(K, \mathbb{R})$ are A-spaces then their Cartesian product

$$M = M_1 \times \cdots \times M_k = \{(q_1, ..., q_k): q_i \in M_i, \, 1 \leqslant i \leqslant k\}$$

is an A-space in $C(K, \mathbb{R}_p^k)$, i.e., it is Chebyshev in $C_\omega(K, \mathbb{R}_p^k)$ for all $\omega \in W$. (Weaker versions of this result can be found in [3] and [4].)

It was proved by Havinson [2] that if $M \subset C([a, b], \mathbb{R})$ is a Chebyshev subspace of $C_\omega([a, b], \mathbb{R})$ for each $\omega \in W$ and elements of M do not vanish on intervals then M is a Haar space at (a, b), i.e., each $q \in M \setminus \{0\}$ has at most dim $M - 1$ zeros at (a, b). (This result can be also deducted from the necessity of the A-property, see [5].) We shall give now the analogue of this statement in the general case.

Consider $f, g \in C(K, X)$. Let us say that f is locally orthogonal to g, written $f \perp_{\text{loc}} g$, if $\tau_X(f, g) = 0$ a.e. at a nonempty open subset of $K \setminus Z(f)$. If $\tau_X(f, g) = 0$ a.e. on the whole set $K \setminus Z(f)$ then we say that f is orthogonal to g, written $f \perp g$. (Note that if X is smooth then $\tau_X(u, v) = 0$ is equivalent to Birkhoff orthogonality of u to v.) As usual an open set $\theta \subset K$ is called r-disconnected if it is a union of r disjoint open sets.

THEOREM 5.   *Let M, dim $M = m$, be a linear subspace of $C(K, X)$, where X is a smooth Banach space. Assume that for any $q_1, q_2 \in M \setminus \{0\}$ the relation $q_1 \perp_{\text{loc}} q_2$ implies $q_1 \perp q_2$. Then if M is a Chebyshev subspace of $C_\omega(K, X)$ for all $\omega \in W$ it follows that $K \setminus Z(q)$ is at most m-disconnected for any $q \in M \setminus \{0\}$.*

Proof. Assume that in contrary for some $q \in M \setminus \{0\}$ we have $K \setminus Z(q) = \bigcup_{j=1}^{m+1} Q_j$, where $Q_j - s$ are nonempty open disjoint sets. Let $\tilde{M}$ be the set of those elements $\tilde{q}$ in M for which $\tau_X(q, \tilde{q}) = 0$ a.e. at $K \setminus Z(q)$. The linearity of τ_X-functional implies that $\tilde{M}$ is a linear subspace of M. Let $q_i, 1 \leqslant i \leqslant m$, be a basis in M such that $q_i, 1 \leqslant i \leqslant r \ (0 \leqslant r \leqslant m - 1)$ is a basis in $\tilde{M}$ and set $M_1 = \operatorname{span}\{q_{r+1}, ..., q_m\}$. Consider the $(m - r) \times (m + 1)$ matrix

$$B(\omega) = \left\{ \int_{Q_j} \omega(x) \, \tau_X(q, q_i)(x) \, d\mu \right\}_{r+1 \leqslant i \leqslant m}^{1 \leqslant j \leqslant m+1}.$$

Furthermore, denote by B^* the set of all those $(m - r) \times (m + 1)$ matrices for which every $(m - r) \times (m - r)$ submatrix has nonzero determinant. It can be easily shown that B^* is a dense subset of $\mathbb{R}^{(m-r)(m+1)}$. In [2] this statement is proved for $r = 0$, the proof for any $0 \leqslant r \leqslant m - 1$ is similar. Assume that for some $\omega \in W$ we have $B(\omega) \in B^*$. Then the linear system of equations

$$\sum_{j=1}^{m+1} a_j \int_{Q_j} \omega(x) \, \tau_X(q, q_i)(x) \, d\mu = 0 \qquad (r + 1 \leqslant i \leqslant m) \tag{14}$$

has solutions $a_j \in \mathbb{R} \setminus \{0\}$, $1 \leqslant j \leqslant m + 1$. Set

$$\omega^*(x) = \begin{cases} |a_j| \, \omega(x), \, x \in Q_j, & 1 \leqslant j \leqslant m + 1 \\ 1, \, x \in Z(q), \end{cases} \tag{15}$$

$\omega^* \in W$, and

$$q^* = \begin{cases} q(x) \operatorname{sign} a_j, & x \in Q_j, 1 \leqslant j \leqslant m + 1 \\ 0, \, x \in Z(q). \end{cases} \tag{16}$$

Evidently, q^* is continuous at K and $q^*(x) = \pm q(x)$ for every $x \in K$. Thus $q^* \in M^* \setminus \{0\}$. Moreover, using again the linearity of τ_X-functional we obtain for $x \in Q_j$ and any $r + 1 \leqslant i \leqslant m$

$$\operatorname{sign} a_j \tau_X(q, q_i)(x) = \tau_X(q, \operatorname{sign} a_j q_i)(x) = \tau_X(\operatorname{sign} a_j q, q_i)(x).$$

Hence and by (15) and (16) we can rewrite (14) as

$$0 = \sum_{j=1}^{m+1} a_j \int_{Q_j} \omega(x) \, \tau_X(q, q_i)(x) \, d\mu$$

$$= \sum_{j=1}^{m+1} \int_{Q_j} \omega^*(x) \, \tau_X(q^*, q_i)(x) \, d\mu$$

$$= \int_{K \setminus Z(q)} \omega^*(x) \, \tau_X(q^*, q_i)(x) \, d\mu \qquad (r + 1 \leqslant i \leqslant m). \tag{17}$$

Finally, using that $\tau_X(q^*, q_i) = \pm\tau_X(q, q_i) = 0$ a.e. at $K \setminus Z(q)$ for every $1 \leqslant i \leqslant r$ we can derive from (17) that

$$\int_{K \setminus Z(q^*)} \omega^*(x) \, \tau_X(q^*, \bar{q})(x) \, d\mu = 0, \qquad \bar{q} \in M.$$

Thus M is not an A-space, which in view of Theorem 3 contradicts the assumptions of our theorem. By this contradiction we obtain that the set $B = \{ B(\omega) : \omega \in W \}$ has empty intersection with B^*. Since B^* is dense in $\mathbb{R}^{(m-r)(m+1)}$ and B is a convex subset of $\mathbb{R}^{(m-r)(m+1)}$ it follows that B has empty interior, i.e., it belongs to hyperplane. Thus for some $c_{i,j} \in \mathbb{R}$ (not all of them zero) we have for every $\omega \in W$

$$0 = \sum_{j=1}^{m+1} \sum_{i=r+1}^{m} c_{ij} \int_{Q_j} \omega(x) \, \tau_X(q, q_i)(x) \, d\mu$$

$$= \sum_{j=1}^{m+1} \int_{Q_j} \omega(x) \, \tau_X(q, \tilde{q}_j)(x) \, d\mu$$

where $\tilde{q}_j \in M_1$, $1 \leqslant j \leqslant m+1$, and at least one of $\tilde{q}_j - s$ is nontrivial. This latter relation yields that $\tau_X(q, \tilde{q}_j) = 0$ a.e. at Q_j, $1 \leqslant j \leqslant m+1$, i.e., if $\tilde{q}_j$ is nontrivial then $q \perp_{\text{loc}} \tilde{q}_j$. But by the assumption of the theorem this implies that $q \perp \tilde{q}_j$, i.e., for some $1 \leqslant j \leqslant m+1$, $\tilde{q}_j \in \tilde{M} \setminus \{0\}$, a contradiction. The theorem is proved.

COROLLARY 2. *Let X be a smooth Banach space and let M be a finite dimensional subspace of $C([a, b], X)$ such that for any $q_1, q_2 \in M \setminus \{0\}$ the relation $q_1 \perp_{\text{loc}} q_2$ implies $q_1 \perp q_2$, and no nontrivial element of M vanishes at an interval. Then if M is a Chebyshev subspace of $C_\omega([a, b], X)$ for all $\omega \in W$ it follows that M is a Haar space at (a, b).*

Let us give an example of application of Theorem 5.

Set $K = [a, b]$, $X = \mathbb{C} = R_2^2$. Note that for $u, v \in \mathbb{C}$ $(u \neq 0)$, $\tau_\mathbb{C}(u, v) = \text{Re } v \, \text{sign } u$, where $\text{sign } u = \bar{u}/|u|$. Let $M_n = \{ \sum_{k=0}^{n} C_k e^{ir_k x} : C_k \in \mathbb{C} \}$ where $0 = r_0 < r_1 < \cdots < r_n$ are integers.

It was shown by Havinson [2] (see also [7] for a more general statement) that if $r_k = k$, $0 \leqslant k \leqslant n$, then M_n is an A-subspace of $C([a, b], \mathbb{C})$ $(0 \leqslant a < b \leqslant 2\pi)$. Let us show that in order that M_n be an A-space it is necessary that M_n be a Haar space at (a, b) $(0 \leqslant a < b \leqslant 2\pi)$. We need only check that M_n satisfies the conditions of Corollary 2. Evidently no element of M_n vanishes at an interval. Furthermore, if for some $q_1, q_2 \in M_n \setminus \{0\}$ we have $\tau_\mathbb{C}(q_1, q_2)(x) = 0$, a.e., at an interval then $\text{Re } \bar{q}_1(x) q_2(x) = 0$ at an interval. But $\text{Re } \bar{q}_1 q_2$ is a real trigonometric polynomial, i.e., $\text{Re } \bar{q}_1 q_2$ must be identically zero. Thus $\tau_\mathbb{C}(q_1, q_2) = 0$ at

$[a, b] \setminus Z(q_1)$. This means that Corollary 2 can be applied for M_n. In particular, we obtain that M_n is an A-subspace of $C([0, 2\pi], \mathbb{C})$ only if $r_k = k$, $k = 1, \dots, n$.

THE A-PROPERTY

In this last section we shall consider the question of existence and characterization of A-spaces. This problem is well studied in the case when $K = [a, b]$, $X = \mathbb{R}$. In this situation a classical example of an A-space is a Haar space at (a, b), i.e., Theorem 2 gives the well-known Jackson–Krein theorem. Some examples of Haar-type A-spaces in the case when $K = [a, b]$, $X = \mathbb{C}(= \mathbb{R}_2^2)$ were given by Kripke and Rivlin [3], Havinson [2] and the author [6]. In a series of papers by Strauss [12, 13, 14], Galkin [1], Sommer [16] and others it was shown that different families of spline functions also satisfy the A-property if $K = [a, b]$, $X = \mathbb{R}$ (see also [4] for the case $X = \mathbb{C}$). Furthermore Sommer [18] proved that A-spaces satisfy the Weak Chebyshev property. These results raised the problem of complete "identification" of the A-spaces. Recently, this problem was solved in the case $K = [a, b]$, $X = \mathbb{R}$ by Pinkus [10]. The essence of his result is that A-spaces are composed piecewise from Haar spaces, i.e., they are generalized splines of a certain type.

Let us turn to the question of A-spaces of real valued functions of several variables. Let us assume that K is in $\mathbb{R}^2$ and $X = \mathbb{R}$. The most natural candidates for A-spaces seem to be algebraic polynomials $P_n = \{\sum_{i+j \leq n} a_{ij} x^i y^j : a_{ij} \in \mathbb{R}\}$. It can be shown (see [8]) that if K is convex than P_1 is indeed an A-space. But, unfortunately, this turns out to be an exception. It is an easy exercise to check that P_n, $n \geq 2$, does not satisfy the A-property if K has nonempty interior in $\mathbb{R}^2$. Analogous remarks hold in connection with the polynomials

$$P_{n,m} = \left\{ \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j : a_{ij} \in \mathbb{R} \right\} \qquad \text{if} \quad n, m \geq 1.$$

Sommer [18] pointed out that linear splines of two variables does not satisfy the A-property, as well. Of course there can be given a trivial example of an A-space in $C(K, \mathbb{R})$, $K \subset \mathbb{R}^2$, of arbitrary dimension n simply by considering a linear span of nonnegative functions with disjoint supports. The above observations indicate that it is very probable that A-spaces of real functions of several variables do not exist apart from some trivial cases. Of course, the situation is different in the complex case, because, for instance, algebraic polynomials satisfy the A-property.

Thus our approach which consisted in studying the uniqueness with

respect to *all* weights is convenient for real functions of only one variable and it seems to become very restrictive if we turn to real functions of several variables. This, of course, does not mean that we can not have nice Chebyshev subspaces with respect to a *single* weight in the real multivariate case. In fact it was shown in [8] that tensor products of Haar spaces of arbitrary dimension with two-dimensional Haar spaces are Chebyshev in $C_1(K_2, \mathbb{R})$ if K_2 is a rectangular region in $\mathbb{R}^2$ (here $\omega \equiv 1$). Thus, in particular, $P_{1,m}$ and $P_{n,1}$ are Chebyshev subspaces of $C_1(K_2, \mathbb{R})$. On the other hand $P_{1,m}$ and $P_{n,1}$ do not satisfy the A-property, i.e., they are not Chebyshev with respect to some other weight. This is another illustration of the fact that the A-property is not necessary in general for uniqueness with respect to a single weight.

Finally we would like to conjecture that $P_{n,m}$ is a Chebyshev subspace of $C_1(K_2, \mathbb{R})$ for any $n, m \geqslant 1$. In [9] we proved a weaker result showing that $P_{n,m}$ is Chebyshev in $P \subset C_1(K_2, \mathbb{R})$, where $P = \bigcup_{i,j} P_{i,j}$.

Remark. By the time the present paper was completed Professor M. Sommer kindly sent the author a preprint in which he verified Theorems 1–3 of this paper in the case $X = \mathbb{R}$. Sommer also presents in his preprint an interesting example of bivariate linear vertex splines which satisfy the A-property.

REFERENCES

1. P. V. GALKIN, The uniqueness of the element of best mean approximation to a continuous function using splines with fixed nodes, *Math. Notes* **15** (1974), 3–8.
2. S. J. HAVINSON, On uniqueness of functions of best approximation in the metric of the space L^1, Izv, Akad. Nauk SSSR **22** (1958), 243–270.
3. B. R. KRIPKE AND T. J. RIVLIN, Approximation in the metric of $L^1(X, \mu)$, *Trans. Amer. Math. Soc.* **119** (1965), 101–122.
4. A. KROÓ, Some theorems on best L_1-approximation of continuous functions, *Acta Math. Hungar* **44** (1984), 409–417.
5. A. KROÓ, On an L_1-approximation problem, *Proc. Amer. Math. Soc.* **94** (1985), 406–410.
6. A. KROÓ, Best L_1-approximation of vector-valued functions, *Acta Math. Hungar.* **39** (1982), 303–310.
7. A. KROÓ, On unicity of complex polynomial L_1-approximation along curves, *Proc. Amer. Math. Soc.* **86** (1982), 427–432.
8. A. KROÓ, Some theorems on unicity of multivariate L_1-approximation, *Acta Math. Hungar.* **40** (1982), 179–189.

9. A. KROÓ, On the unicity of best L_1-approximation by polynomials of several variables, *Acta Math. Hungar.* **42** (1983), 309–318.

10. A. PINKUS, Unicity subspaces in L^1-approximation, *J. Approx. Theory*, in press.

11. H. STRAUSS, Uniqueness in L_1-approximation for continuous functions, in "Approximation Theory III" (E. W. Cheney, Ed.), 865–870, Academic Press, New York, 1980.

12. H. STRAUSS, Eindeutigkeit in der L_1-approximation, *Math. Z.* **176** (1981), 63–74.

13. H. STRAUSS, Best L_1-approximation, *J. Approx. Theory* **41** (1984), 297–308.

14. H. STRAUSS, L_1-approximation mit splinefunktionen, in "Numerische Methoden der Approximations theorie" (L. Collatz and G. Meinardus, Eds.), 151–162, ISNM Vol. 26, Birkhäuser–Verlag, Basel and Stuttgart.

15. E. ROSEMA, Almost Chebyshev subspaces of $L^1(\mu, E)$, *Pacific J. Math.* **53** (1974), 585–604.

16. M. SOMMER, L_1-approximation by weak Chebyshev spaces, in "Approximation in Theorie und Praxis," (G. Meinardus, Ed.) 85–102, Bibliographises Institut, Mannheim, 1979.

17. M. SOMMER, Weak Chebyshev spaces and best L_1-approximation, *J. Approx. Theory* **39** (1983), 54–71.

18. M. SOMMER, Some results on best L_1-approximation of continuous functions, *Numer. Funct. Anal. Optim.* **6** (1983), 253–271.